Algebraic Geometry FINAL

There are 6 questions. Questions 1,2,5 and 6 are 10 points each. Questions 3 and 4 are 5 points each. Please look over the entire paper before attempting to answer as some questions may be easier than others.

1. Show that the set $GL_n(\mathbb{C})$ of invertible $n \times n$ matrices has the structure of an affine algebraic variety.
2. Fix an irreducible conic C in $\mathbb{P}^2$. Show that the set of lines that fail to meet C in exactly two points is a closed subvariety of the Grassmanian of lines in $\mathbb{P}^2$, $Gr(2,3)$.
3. Compute the Hilbert Polynomial of $\mathbb{P}^3$.
4. The function field $\mathbb{C}(X)$ of an irreducible variety X is the fraction field of its co-ordinate ring of an affine open subset. Show that two irreducible varieties X and Y are birationally equivalent if and only if $\mathbb{C}(X) \simeq \mathbb{C}(Y)$.
5. An algebraic variety is said to be *rational* if it is birationally equivalent to projective space.

   i. Show that the nodal curve

$$Y^2 - X^2 - X^3 = 0$$

is rational.

   ii. Show that $\mathbb{P}^1 \times \mathbb{P}^1$ is rational.

6. Let $X \subset \mathbb{A}^{n+1}$ be the affine cone over a smooth projective variety. Show that the cone can be desingularized by blowing up the vertex. What is the fibre over this vertex in the desingularization ?